Alignment and algebraically special tensors in Lorentzian geometry.

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We develop a dimension-independent theory of alignment in Lorentzian geometry, and apply it to the tensor classification problem for the Weyl and Ricci tensors. First, we show that the alignment condition is equivalent to the PND equation. In 4D, this recovers the usual Petrov types. For higher dimensions, we prove that, in general, a Weyl tensor does not possess aligned directions. We then go on to describe a number of additional algebraic types for the various alignment configurations. For the case of second-order symmetric (Ricci) tensors, we perform the classification by considering the geometric properties of the corresponding alignment variety.

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I. INTRODUCTION

In this paper we study the notion of alignment in Lorentzian geometry, and apply our results to the problem of tensor classification. Our ideas are inspired by and generalize covariant classification methods that are utilized in general relativity. In four dimensions, tensor classification is important for physical applications and, in particular, for the study of exact solutions of the Einstein equations [1]. Beyond the classical theory, there has been great interest in higher dimensional Lorentz manifolds as models for generalized field theories that incorporate gravity [2]. The problem of tensor classification in higher dimensions [3, 4] is therefore of interest.

The most important algebraic classification results in four-dimensional (4D) general rel-

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ativity are the classification of the energy-momentum tensor according to Segre type, and the classification of the Weyl tensor according to Petrov type. Our motivation is a general scheme applicable to the classification of arbitrary tensor types in arbitrarily high dimensions. Thus, after developing a general theory of alignment, we apply our results to the classification problem of higher-dimensional Weyl tensors. We also use alignment to re-derive and extend classification results for second-order symmetric tensors.

Classification of algebraic tensor types in Lorentzian geometry has a purely mathematical importance. The indefinite signature makes Lorentzian geometry profoundly different from Riemannian geometry. For example, a second-order symmetric tensor need not be diagonalizable when the signature is indefinite. Another striking difference was recently brought to light in a classification of Lorentzian 4D manifolds characterized by the property that all scalar curvature invariants vanish (VSI spacetimes)[5]. In Riemannian geometry, such a manifold must necessarily be flat, but this is not so in Lorentzian geometry, where there is a wealth of non-flat examples. Thus, understanding of algebraic tensor types in Lorentzian geometry is needed for the continued investigation of Lorentzian geometric phenomena, without analogues in Riemannian geometry.

In practice, a complete tensor classification is possible only for simple symmetry types and for small dimensions n. However, partial classification into broader categories is still desirable. One wants a classification framework that is as general as possible but, at the same time, sufficiently detailed so as to reproduce prior classifications of specific symmetry types and dimensions.

Penrose and Rindler[6, Ch. 8] take a step in this direction by describing a framework for the classification of maximally symmetric tensors and spinors in four dimensions. Their classification works by associating an algebraic curve on 2-dimensional extended space (the 2-sphere) to a given symmetric spinor, and then utilizing covariant, algebro-geometric properties of the curve, such as factor multiplicities and singular points, to classify the spinor.

The point of departure for the present article is the observation that the Penrose-Rindler approach can be reinterpreted and generalized in terms of frame fixing. The components of a Lorentzian tensor can be naturally ordered according to boost weight. In essence, one counts the $\boldsymbol{n}$ of the NP tetrad with weight 1, the $\boldsymbol{\ell}$ with weight -1, and the space-like components with weight 0. We will call a null direction $\boldsymbol{\ell}$ aligned with the tensor, if the components with the largest weight vanish along that direction. The Penrose-Rindler curve is just the locus of aligned null directions. Singular points of the curve correspond to null directions of higher alignment, ones for which multiple leading orders vanish.

Thus, the key concept in our theory is the notion of an aligned null direction and of alignment order. We therefore begin by showing that the set of null directions aligned with a fixed n-dimensional tensor is a variety (we call it the top alignment variety), meaning that it can be described by a certain set of polynomial equations in $n-2$ variables. Covariance now becomes a key issue, because these alignment equations are given with respect to a particular null-frame. A change of null-frame transforms the equations in a covariant fashion. The necessary mathematical framework needed to describe such covariant compatibility is a scheme, a very general notion from algebraic geometry that we adapt to the present circumstances.

The null directions of higher alignment order are subvarieties of the top alignment variety. These higher-order directions have an important geometric characterization related to singularities. Indeed, we prove that for irreducible representations of the Lorentz group, the equations for higher order alignment are equivalent to the equations for the singular points

of the top alignment variety. Thus, for tensors, such as trace-free symmetric R_{ab}, bivectors, and Weyl tensors — these all belong to irreducible representations — the algebraically special tensors can be characterized as the instances whose alignment variety is singular.

Having dispensed with the necessary mathematical preliminaries, we describe a covariant classification methodology based on alignment order. In essence, we fix a null-frame $\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_i$ so that the order of alignment of $\boldsymbol{\ell}$ and $\boldsymbol{n}$ along the given tensor is as large as possible. We then define the alignment type of a tensor to be the alignment order of such $\boldsymbol{\ell}$ and $\boldsymbol{n}$.

We illustrate the alignment classification with examples of vectors, four-dimensional bivectors, second-order symmetric tensors, and Weyl-like, valence four tensors. We show how to re-derive the known, four-dimensional classification results for these tensors — the Segre and Petrov types. We also consider the classification of these tensor types in higher-dimensions.

For symmetric R_{ab}, the higher dimensional classification is a straight-forward generalization of the situation in four dimensions. The situation for higher-dimensional Weyl tensors is more complicated. We prove that the well-known homogeneous PND equations[1, 7], generalized to arbitrary dimensions, are equivalent to the equations of alignment. However, we prove that for $n > 4$, these equations do not, generically, possess a solution. Thus, in contrast to 4D where an aligned $\boldsymbol{\ell}$ and an aligned $\boldsymbol{n}$ can always be found, in higher dimensions it is necessary to introduce new Weyl types to account for the possibility that there does not exist an aligned $\boldsymbol{\ell}, \boldsymbol{n}$. The end result of this analysis is a coarse classification, in the sense that it is fully equivalent to the Petrov classification in four dimension, but does not always lead to canonical forms for higher-dimensional Weyl tensors.

II. PRELIMINARIES.

A. Lorentz and Möbius geometry.

Our setting is n-dimensional Minkowski space. We define this to be a vector space isomorphic to $\mathbb{R}^n$, together with a Lorentz-signature inner product g_{ab}. We define a *null frame* to be a basis $\boldsymbol{\ell} = \boldsymbol{m}_0$, $\boldsymbol{n} = \boldsymbol{m}_1$, $\boldsymbol{m}_2, \ldots, \boldsymbol{m}_{n-1}$, satisfying $\ell^a n_a = 1$, ${m_i}^a m_{ja} = \delta_{ij}$, with all other products vanishing. In accordance with the above signature convention, a vector $\mathbf{u}$ will be called space-like, time-like, or null depending on whether the norm $u^a u_a$ is respectively, positive, negative, or zero.

Throughout, roman indices $a, b, c, \mathrm{A}, \mathrm{B}, \mathrm{C}$ range from 0 to $n-1$. Lower case indices indicate an arbitrary basis, while the upper-case ones indicate a null frame. Space-like indices i, j, k also indicate a null-frame, but vary from 2 to $n-1$ only. We raise and lower the space-like indices using δ_{ij}, so that $\boldsymbol{m}_i = \boldsymbol{m}^i$. The Einstein summation convention is observed throughout.

We let $\Phi^{\mathrm{A}}{}_{\mathrm{B}}$ denote a null-frame orthogonal matrix, and characterize a Lorentz transformation as a change of null-frame, $\hat{\boldsymbol{m}}_{\mathrm{B}} = \boldsymbol{m}_{\mathrm{A}} \Phi^{\mathrm{A}}{}_{\mathrm{B}}$. The group of orthochronous[8] Lorentz transformations is generated by null rotations (1), boosts (2), and spins (3), which are transformations of form

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell} + z^j \boldsymbol{m}_j - \tfrac{1}{2} z^j z_j \, \boldsymbol{n}, \quad \hat{\boldsymbol{n}} = \boldsymbol{n}, \quad \hat{\boldsymbol{m}}_i = \boldsymbol{m}_i - z_i \boldsymbol{n}; \tag{1}$$

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \quad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \quad \hat{\boldsymbol{m}}_i = \boldsymbol{m}_i, \quad \lambda \neq 0; \tag{2}$$

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell}, \quad \hat{\boldsymbol{n}} = \boldsymbol{n}, \quad \hat{\boldsymbol{m}}_j = \boldsymbol{m}_i X^i{}_j, \quad X^i{}_j X^j_k = \delta^i{}_k. \tag{3}$$

The following matrix represents a null-rotation about $\boldsymbol{n}$:

$$\Lambda^{\mathrm{A}}{}_{\mathrm{B}}(z_i) = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2}z^j z_j & 1 & -z_i \\ z^j & 0 & \delta^j{}_i \end{pmatrix} \tag{4}$$

The set of null lines in n-dimensional Minkowski space is an m-dimensional variety

$$\mathbb{PK}^m = \{[\boldsymbol{k}] : k^a k_a = 2k_0 k_1 + k_i k^i = 0\}, \quad m = n-2.$$

Affine coordinates $z_i = k_i/k_1$ are defined for every choice of null-frame. Over the real field, we regard $[\boldsymbol{n}]$ as a point at infinity, and identify $\mathbb{RPK}^m$ with real extended space $\hat{\mathbb{R}}^m = \mathbb{R}^m \cup \{\infty\}$, the one point compactification of $\mathbb{R}^m$ homeomorphic to the sphere $\mathbb{S}^m$. We will also need to consider the complexification of extended space

$$\hat{\mathbb{C}}^m = \mathbb{CPK}^m = \mathbb{C}^m \cup \mathbb{CK}^{m-1} \cup \hat{\mathbb{C}}^{m-2},$$

where the elements of the second term and third terms are points at infinity having, respectively, the form $[z^i \boldsymbol{m}_i + \boldsymbol{n}]$, and $[z^i \boldsymbol{m}_i]$, with $z^i z_i = 0$.

A Lorentz transformation $\hat{\boldsymbol{m}}_{\mathrm{B}} = \boldsymbol{m}_{\mathrm{A}} \Phi^{\mathrm{A}}{}_{\mathrm{B}}$ induces a birational transformation of extended space, described by

$$z^j = \frac{\phi^j(\hat{z}_i)}{\phi^0(\hat{z}_i)}, \quad \phi^{\mathrm{A}}(\hat{z}_i) = \Phi^{\mathrm{A}}{}_0 + \Phi^{\mathrm{A}}{}_i\, \hat{z}^i - \tfrac{1}{2}\Phi^{\mathrm{A}}{}_1\, \hat{z}^i \hat{z}_i. \tag{5}$$

This can be seen by noting that

$$\hat{\boldsymbol{\ell}} - \tfrac{1}{2}\hat{z}^i \hat{z}_i \hat{\boldsymbol{n}} + \hat{z}^i \hat{\boldsymbol{m}}_i = \phi^0(\hat{z}_i)\boldsymbol{\ell} + \phi^1(\hat{z}_i)\boldsymbol{n} + \phi^j(\hat{z}_i)\boldsymbol{m}_j\,. \tag{6}$$

A real transformation of form (5) is a conformal transformation of $\mathbb{S}^m$, and is known as a Möbius transformation [9, 10]. In terms of the affine coordinates, null rotations about $\boldsymbol{n}$ correspond to translations; null rotations about $\boldsymbol{\ell}$ to inversions; boosts correspond to dilations; spins correspond to rotations. The infinitesimal generators for these transformations are listed in Table I.

TABLE I: infinitesimal generators

Generator	Lorentz transformation	Möbius transformation
∂_i	null rotation about $\boldsymbol{n}$	translation
$z^j \partial_j$	boost	dilation
$z^i \partial_j - z^j \partial_i$	spin	rotation
$z^i z^j \partial_j - \frac{1}{2} z^j z_j \partial_i$	null rotation about $\boldsymbol{\ell}$	inversion

B. Möbius schemes and subvarieties.

Let $\mathbb{C}[z_i]$ denote the ring of polynomials in the m indeterminates z_i. For an affine ideal, $I \subset \mathbb{C}[z_i]$, we let

$$\mathbf{V} = \mathbf{V}(I) = \{\zeta^i \in \mathbb{C}^m : p(\zeta^i) = 0, \quad p(z^i) \in I\}$$

denote the corresponding affine variety. Tensor components are defined relative to a particular choice of null-frame, and transform covariantly with respect to (8). In order to treat varieties embedded in extended space, we need to define a similar notion of covariance for affine ideals.

For $q(z_i) \in \mathbb{C}[z_i]$, the fraction ring $\mathbb{C}[z_i, q^{-1}] = \{p(z_i)/q(z_i)^n : p \in \mathbb{C}[z_i]\}$ is called the *localization* by q. It represents restriction of a polynomial equations to the domain $q(z_i) \neq 0$. We let $I_q = \{p(z_i)/q(z_i)^d : p \in I\}$ denote the corresponding localization of an affine ideal I.

Let $\hat{\boldsymbol{m}}_{\mathrm{B}} = \boldsymbol{m}_{\mathrm{A}} \Phi^{\mathrm{A}}{}_{\mathrm{B}}$ be null-frames related by a Lorentz transformation. Let $\Psi^{\mathrm{A}}{}_{\mathrm{B}}$ be the inverse transformation, and let

$$z^j = \frac{\phi^j(\hat{z}_i)}{\phi^0(\hat{z}_i)}, \quad \hat{z}^j = \frac{\psi^j(z_i)}{\psi^0(z_i)}, \tag{7}$$

be the corresponding inverse Möbius transformations.

Definition II.1 *We say that affine ideals $I \subset \mathbb{C}[z_i]$, $\hat{I} \subset \mathbb{C}[\hat{z}_i]$ are* covariantly compatible *if the localized ideals I_{ϕ_0}, $\hat{I}_{\psi^0}$ are related by the substitutions* (7)*. We define a* Möbius scheme *$\mathcal{A}$ to be a covariantly compatible assignment of an affine ideal $A \subset \mathbb{C}[z_i]$ to every null frame $\boldsymbol{m}_{\mathrm{A}}$.*

We should note that our notion of a scheme is a simplified adaptation of the usual definition of such objects from algebraic geometry[11].

Definition II.2 *We define $\mathbf{V}(\mathcal{A}) \subset \hat{\mathbb{C}}^m$, the* Möbius subvariety *corresponding to $\mathcal{A}$, to be the union of affine varieties*

$$\mathbf{V}(\mathcal{A}) = \bigcup \{[\boldsymbol{\ell} - \tfrac{1}{2}\zeta^i\zeta_i \boldsymbol{n} + \zeta^i \boldsymbol{m}_i] \in \hat{\mathbb{C}}^m : p(\zeta_i) = 0, \text{ for all } p(z_i) \in A\},$$

where the union is taken over all possible null frames $\boldsymbol{m}_{\mathrm{A}}$.

III. ALIGNMENT

A. Boost order.

Let $\mathbf{T} = T_{a_1 \dots a_p}$ be a rank p tensor. For a given list of frame indices $\mathrm{A}_1, \dots, \mathrm{A}_p$, we call the corresponding $T_{\mathrm{A}_1 \dots \mathrm{A}_p}$ a *null-frame scalar*. A Lorentz transformation $\Phi^{\mathrm{A}}{}_{\mathrm{B}}$ transforms the scalars according to

$$\hat{T}_{\mathrm{B}_1 \dots \mathrm{B}_p} = T_{\mathrm{A}_1 \dots \mathrm{A}_p} \Phi^{\mathrm{A}_1}{}_{\mathrm{B}_1} \cdots \Phi^{\mathrm{A}_p}{}_{\mathrm{B}_p}. \tag{8}$$

In particular, a boost (2) transforms the scalars according to:

$$\hat{T}_{\mathrm{A}_1 \dots \mathrm{A}_p} = \lambda^{\mathrm{b}_{\mathrm{A}_1 \dots \mathrm{A}_p}} T_{\mathrm{A}_1 \dots \mathrm{A}_p}, \quad \mathrm{b}_{\mathrm{A}_1 \dots \mathrm{A}_p} = \mathrm{b}_{\mathrm{A}_1} + \dots + \mathrm{b}_{\mathrm{A}_p}, \tag{9}$$

where $\mathrm{b}_0 = 1$, $\mathrm{b}_i = 0$, $\mathrm{b}_1 = -1$.

Definition III.1 *We will call* $\mathrm{b}_{\mathrm{A}_1\ldots\mathrm{A}_p}$ the boost weight *of the scalar* $T_{\mathrm{A}_1\ldots\mathrm{A}_p}$. *Equivalently, the boost weight of* $T_{\mathrm{A}_1\ldots\mathrm{A}_p}$ *is the difference between the number of subscripts equal to* 0 *and the number of subscripts equal to* 1. *We define the* boost order *of the tensor* $\mathbf{T}$, *as a whole, to be the boost weight of its leading term, or to put it another way, the maximum* $\mathrm{b}_{\mathrm{A}_1\ldots\mathrm{A}_p}$ *for all non-vanishing* $T_{\mathrm{A}_1\ldots\mathrm{A}_p}$.

Proposition III.2 *Let* $\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_i$ *and* $\hat{\boldsymbol{\ell}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{m}}_i$ *be two null-frames with* $\boldsymbol{\ell}$ *and* $\hat{\boldsymbol{\ell}}$ *scalar multiples of each other. Then, the boost order of a given tensor is the same relative to both frames.*

Proof. A null rotation about $\boldsymbol{\ell}$ fixes the leading terms of a tensor, while boosts and spins (9) (3) subject the leading terms to an invertible transformation. It follows that the boost order of a tensor, as a whole, does not depend on a choice of a particular null-frame, but rather on the choice of $\boldsymbol{\ell}$. QED

Definition III.3 *Let* $\mathbf{T}$ *be a tensor, and* $[\boldsymbol{k}]$, $k^a k_a = 0$ *a null direction. We choose a null frame* $\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_i$, *such that* $\boldsymbol{\ell}$ *is a scalar multiple of* $\boldsymbol{k}$, *and define* $\mathrm{b}_{\mathbf{T}}(\boldsymbol{k})$, *the* boost order along $\boldsymbol{k}$, *to be the boost order of* $\mathbf{T}$ *relative to this frame.*

This definition is sound, because, by the preceding Proposition, the boost order is the same for all such frames. Usually, the choice of tensor $\mathbf{T}$ is clear from the context, and so we suppress the subscript and simply write $\mathrm{b}(\boldsymbol{k})$.

Definition III.4 *We let* $\mathrm{b}_{\max}$ *denote the maximum value of* $\mathrm{b}(\boldsymbol{k})$ *taken over all null vectors* $\boldsymbol{k}$, *and say that a null vector* $\boldsymbol{k}$ *is* aligned *with the tensor* $\mathbf{T}$ *whenever* $\mathrm{b}(\boldsymbol{k}) < \mathrm{b}_{\max}$. *We will call the integer* $\mathrm{b}_{\max} - \mathrm{b}(\boldsymbol{k}) - 1 \geq 0$ *the order of the alignment.*

The value of $\mathrm{b}_{\max}$ depends on the rank and on the symmetry properties of the tensor $\mathbf{T}$. Generically, for a rank p tensor we have $\mathrm{b}_{\max} = p$. However, if the tensor has some skew-symmetry, then $\mathrm{b}_{\max}$ will be smaller than p. For example, the boost weights for a second-order, symmetric tensor $R_{ab} = R_{ba}$ are shown below. Thus, $\mathrm{b}(\boldsymbol{\ell}) = 1$ if $R_{00} = 0$ but some $R_{0i} \neq 0$; while $\mathrm{b}(\boldsymbol{\ell}) = 0$ if $R_{00} = R_{0i} = 0$, and one of $R_{10}, R_{ij} \neq 0$; etc.

$$R_{ab} = \overbrace{R_{00}\, n_a n_b}^{2} + \overbrace{2R_{0i}\, n_{(a} m^i{}_{b)}}^{1} + \overbrace{2R_{01}\, \ell_{(a} n_{b)} + R_{ij}\, m^i{}_a m^j{}_b}^{0} + \overbrace{2R_{1i}\, \ell_{(a} m^i{}_{b)}}^{-1} + \overbrace{R_{11}\, \ell_a \ell_b}^{-2}.$$

For a bivector $K_{ab} = -K_{ba}$, we have $\mathrm{b}_{\max} = 1$; the corresponding boost weights are shown below. If $K_{0i} = 0$, then $\boldsymbol{\ell}$ is aligned. If in addition, $K_{01} = K_{ij} = 0$, then the order of alignment is 1.

$$K_{ab} = \overbrace{2K_{0i}\, n_{[a} m^i{}_{b]}}^{1} + \overbrace{2K_{01}\, n_{[a} \ell_{b]} + K_{ij}\, m^i{}_{[a} m^j{}_{b]}}^{0} + \overbrace{2K_{1i}\, \ell_{[a} m^i{}_{b]}}^{-1}. \tag{10}$$

B. The alignment variety

We now show that the set of aligned directions is a Möbius variety by exhibiting a set of compatible equations for this set. Let $\mathbf{T}$ be a rank p tensor and $\boldsymbol{m}_{\mathrm{A}}$ a null-frame. For every choice of indices $\mathrm{A}_1, \ldots, \mathrm{A}_p$ we define the polynomial

$$\tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p}(z_i) = T_{\mathrm{B}_1\ldots\mathrm{B}_p}\, \Lambda^{\mathrm{B}_1}{}_{\mathrm{A}_1}(z_i) \cdots \Lambda^{\mathrm{B}_p}{}_{\mathrm{A}_p}(z_i). \tag{11}$$

where $\Lambda^{\mathrm{B}}{}_{\mathrm{A}}(z_i)$ is the null rotation matrix, defined in (4), parameterized by complex indeterminates z_i.

By Definition III.4, a null vector $\boldsymbol{\ell} - \frac{1}{2}\zeta^i\zeta_i\boldsymbol{n} + \zeta^i\boldsymbol{m}_i$, is aligned with $\mathbf{T}$, with alignment order q, if and only if $z_i = \zeta_i$ satisfies the corresponding q^{th} order *alignment equations*

$$\tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p}(z_i) = 0, \quad \mathrm{b}_{\mathrm{A}_1\ldots\mathrm{A}_p} = \mathrm{b}_{\max} - r, \quad r = 0, 1, \ldots, q. \tag{12}$$

We call the ideal generated by the above polynomials,

$$A^q = A^q(T) = \langle \tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p}(z_i)\rangle, \quad \mathrm{b}_{\mathrm{A}_1\ldots\mathrm{A}_p} \geq \mathrm{b}_{\max} - q,$$

the *alignment ideal* of order q.

Theorem III.5 *Let $\hat{\boldsymbol{m}}_{\mathrm{B}} = \boldsymbol{m}_{\mathrm{A}}\Phi^{\mathrm{A}}{}_{\mathrm{B}}$ be two complex null frames related by Lorentz transformation. Then, the corresponding alignment ideals $A^q, \hat{A}^q$ are covariantly compatible in the sense of Definition II.1.*

Proof. Let $\hat{T}_{\mathrm{B}_1\ldots\mathrm{B}_p}$ be the transformed scalars, as per (8), and let

$$\breve{T}_{\mathrm{A}_1\ldots\mathrm{A}_p}(\hat{z}_i) = \hat{T}_{\mathrm{B}_1\ldots\mathrm{B}_p}\, \Lambda^{\mathrm{B}_1}{}_{\mathrm{A}_1}(\hat{z}_i)\cdots\Lambda^{\mathrm{B}_p}{}_{\mathrm{A}_p}(\hat{z}_i) \tag{13}$$

be the generators of $\hat{A}^q$. Effecting substitutions (7) as necessary, we have

$$\Phi^{\mathrm{C}}{}_{\mathrm{B}}\Lambda^{\mathrm{B}}{}_{\mathrm{A}}(\hat{z}_i) = \Lambda^{\mathrm{C}}{}_{\mathrm{B}}(z_i)\Upsilon^{\mathrm{B}}{}_{\mathrm{A}}(\hat{z}_i), \tag{14}$$

where

$$\Upsilon^{\mathrm{B}}{}_{\mathrm{A}}(\hat{z}_i) = \begin{pmatrix} \phi^0 & \Phi^0{}_1 & \phi^0{}_{,i} \\ 0 & \dfrac{1}{\phi^0} & 0 \\ 0 & \Phi^j{}_1 - \Phi^0{}_1\dfrac{\phi^j}{\phi^0} & \phi^j{}_{,i} - \phi^0{}_{,i}\dfrac{\phi^j}{\phi^0} \end{pmatrix}, \tag{15}$$

and where $\phi^{\mathrm{A}} = \phi^{\mathrm{A}}(\hat{z}_i)$ as per eq. (5).

To prove that $\Upsilon^{\mathrm{B}}{}_{\mathrm{A}}$ has the form show in (15), we set $\breve{\boldsymbol{m}}_{\mathrm{A}} = \hat{\boldsymbol{m}}_{\mathrm{B}}\,\Lambda^{\mathrm{B}}{}_{\mathrm{A}}(\hat{z}_i)$, $\tilde{\boldsymbol{m}}_{\mathrm{A}} = \boldsymbol{m}_{\mathrm{B}}\,\Lambda^{\mathrm{B}}{}_{\mathrm{A}}(z_i)$, and note that (14) is equivalent to $\breve{\boldsymbol{m}}_{\mathrm{A}} = \tilde{\boldsymbol{m}}_{\mathrm{B}}\Upsilon^{\mathrm{B}}_{\mathrm{A}}$. By (6), we have that $\breve{\boldsymbol{\ell}} = \phi^0\,\tilde{\boldsymbol{\ell}}$, giving us the first column of $\Upsilon^{\mathrm{B}}_{\mathrm{A}}$. To verify the form of the second column, we note that $\breve{\boldsymbol{n}} = \hat{\boldsymbol{n}}$ and use the fact that

$$\breve{n}^a\tilde{\ell}_a = \frac{1}{\phi^0}\,\breve{n}^a\breve{\ell}_a = \frac{1}{\phi^0}.$$

By (5) we have that $\breve{\boldsymbol{m}}_j = \boldsymbol{m}_{\mathrm{A}}\phi^{\mathrm{A}}{}_{,j}(\hat{z}_i)$. Using this, as well as the fact that

$$\breve{m}_j{}^a\tilde{\ell}_a = \frac{1}{\phi^0}\,\breve{m}_j{}^a\breve{\ell}_a = 0,$$

gives the form of the third column.

Returning to the proof of the Proposition, from (8) (11) (13) we have

$$\breve{T}_{\mathrm{B}_1\ldots\mathrm{B}_p} = \tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p}\Upsilon^{\mathrm{A}_1}{}_{\mathrm{B}_1}\cdots\Upsilon^{\mathrm{A}_p}{}_{\mathrm{B}_p}\,. \tag{16}$$

By inspection of (15) we see that a given $\breve{T}_{\mathrm{B}_1\ldots\mathrm{B}_p}$ is a linear combination of $\tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p}$, $\mathrm{b}_{\mathrm{A}_1\ldots\mathrm{A}_p} \geq \mathrm{b}_{\mathrm{B}_1\ldots\mathrm{B}_p}$ with coefficients that are polynomials in $\mathbb{C}[z^i, (\phi^0)^{-1}]$. It follows that the localization of A^q is equal to the localization of $\hat{A}^q$. QED

Definition III.6 *We define $\mathcal{A}^q = \mathcal{A}^q(\mathbf{T})$, the alignment scheme of order q, to be the scheme generated by the alignment ideals $A^q(\mathbf{T})$. The top alignment scheme $\mathcal{A}^0$, generated by equations $\tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p}(z_i) = 0,\ \mathrm{b}_{\mathrm{A}_1\ldots\mathrm{A}_p} = \mathrm{b}_{\max}$ has a distinguished role, and will be denoted simply as $\mathcal{A} = \mathcal{A}(\mathbf{T})$.*

The corresponding varieties $\mathbf{V}(\mathcal{A}^q)$ consist of aligned null directions of alignment order q or more. It may well happen that the alignment equations are over-determined and do not admit a solution, in which case the variety is the empty set.

Definition III.7 *In those cases where $\mathbf{V}(\mathcal{A}^q)$ consists of only a finite number of null directions, we will call these directions* principal *and speak of the* principal null directions *(PNDs) of the tensor.*

C. Singular points and higher alignment.

Definition III.8 *Let I be an affine ideal. A point of the variety $\zeta_i \in \mathbf{V}(I)$ will be called* singular *if the first-order partial derivatives of the polynomials in I vanish at ζ_i also. It will be called singular of order q, if all partials of order q and lower, vanish at that point, i.e., $p_{,j_1\ldots j_r}(\zeta_i) = 0$, $p(z_i) \in I$, $r \leq q$.*

Geometrically, a singular point represents a self-intersection such as a a node or a cusp, or a point of higher multiplicity.

Proposition III.9 *Let $\mathbf{T}$ be a rank p tensor, and $\mathcal{A}$ the corresponding alignment scheme. Suppose that $\boldsymbol{k} = \boldsymbol{\ell} - \frac{1}{2}\zeta^i\zeta_i\boldsymbol{n} + \zeta^i\boldsymbol{m}_i$ spans an aligned direction of order q, i.e., $z_i = \zeta_i$ satisfies* (12). *Then, $[\boldsymbol{k}]$ is a q^{th} order singular point of the top variety $\mathbf{V}(\mathcal{A})$. In other words, $z_i = \zeta_i$ also satisfies all equations of the form*

$$\tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p,i_1\ldots i_r}(z_i) = 0, \quad \mathrm{b}_{\mathrm{A}_1\ldots\mathrm{A}_p} = \mathrm{b}_{\max}, \quad r = 0, 1, \ldots, q. \tag{17}$$

Proof. From (4) we have

$$\Lambda^{\mathrm{A}}{}_{0,i} = \Lambda^{\mathrm{A}}{}_i \quad \Lambda^{\mathrm{A}}{}_{j,i} = -\delta_{ij}\Lambda^{\mathrm{A}}{}_1, \quad \Lambda^{\mathrm{A}}{}_{1,i} = 0. \tag{18}$$

Hence, by (11)

$$\tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p,i} = \tilde{T}_{\mathrm{B}\mathrm{A}_2\ldots\mathrm{A}_p}\lambda^{\mathrm{B}}{}_{\mathrm{A}_1 i} + \tilde{T}_{\mathrm{A}_1\mathrm{B}\ldots\mathrm{A}_p}\lambda^{\mathrm{B}}{}_{\mathrm{A}_2 i} + \ldots + \tilde{T}_{\mathrm{A}_1\mathrm{A}_2\ldots\mathrm{B}}\lambda^{\mathrm{B}}{}_{\mathrm{A}_p i}, \tag{19}$$

where $\lambda^j{}_{0i} = \delta^j{}_i$, $\lambda^1{}_{ji} = -\delta_{ij}$, with all other entries zero. Hence, by (9) the boost weight of the terms in the right hand side of (19) is exactly one smaller than $\mathrm{b}_{\mathrm{A}_1\ldots\mathrm{A}_p}$. Similarly, every r^{th} order partial derivative of $\tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p}(z_i)$ is a linear combination of polynomials $\tilde{T}_{\mathrm{B}_1\ldots\mathrm{B}_p}(z_i)$ for which $\mathrm{b}_{\mathrm{B}_1\ldots\mathrm{B}_p} = \mathrm{b}_{\mathrm{A}_1\ldots\mathrm{A}_p} - r$. If $[\boldsymbol{k}]$ is aligned, with alignment order q, then $z_i = \zeta_i$ satisfies (12). Hence, by the preceding remark, $z_i = \zeta_i$ satisfies (17) also. QED
A partial converse of the preceding is the following.

Proposition III.10 *Suppose that $\mathbf{T}$ belongs to an irreducible representation of the Lorentz group. Let $\boldsymbol{k} = \boldsymbol{\ell} - \frac{1}{2}\zeta^i\zeta_i\boldsymbol{n} + \zeta^i\boldsymbol{m}_i$ be a q^{th}-order singular element of the top alignment variety, i.e., $z_i = \zeta_i$ satisfies* (17). *Then, $\boldsymbol{k}$ spans a q^{th} order aligned null direction, i.e. $z_i = \zeta_i$ satisfies* (12) *also.*

Proof. The polynomials $\tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p}(z_i)$ span an irreducible representation of the Möbius group. The group action is shown in (16). The corresponding infinitesimal generators are matrix differential operators, $\mathbf{D} + M^{\mathrm{B}}{}_{\mathrm{A}}$ where $\mathbf{D}$ is an infinitesimal Möbius transformation, shown in Table I, and where $M^{\mathrm{B}}{}_{\mathrm{A}}$ is the infinitesimal form of $\Upsilon^{\mathrm{A}}{}_{\mathrm{B}}$. These operators act by

$$\tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p} \mapsto \mathbf{D}[\tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p}] + \tilde{T}_{\mathrm{B}\mathrm{A}_2\ldots\mathrm{A}_p}\, M^{\mathrm{B}}{}_{\mathrm{A}_1} + \tilde{T}_{\mathrm{A}_1\mathrm{B}\ldots\mathrm{A}_p}\, M^{\mathrm{B}}{}_{\mathrm{A}_2} + \ldots + \tilde{T}_{\mathrm{A}_1\mathrm{A}_2\ldots\mathrm{B}}\, M^{\mathrm{B}}{}_{\mathrm{A}_p},$$

The operators corresponding to, respectively, null rotations about $\boldsymbol{n}$, boosts, spins, and null rotations about $\boldsymbol{\ell}$, have the form

$$\partial_i, \quad z^j\partial_j + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad z^i\partial_j - z^j\partial_i, \quad z^k z^j \partial_j - \tfrac{1}{2} z^j z_j \partial_k + \begin{pmatrix} -z_k & 0 & -\delta_{ik} \\ 0 & z_k & 0 \\ 0 & \delta^j{}_k & -\delta^j{}_k z_i \end{pmatrix}.$$

The polynomials of maximal boost weight, $\tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p}(z_i)$, $\mathrm{b}_{\mathrm{A}_1\ldots\mathrm{A}_p} = \mathrm{b}_{\max}$, are annihilated by the raising operators and are preserved by the infinitesimal spins and boosts. The lowering operators correspond to partial derivatives with respect to the z_k. Hence, the partial derivatives

$$\tilde{T}_{\mathrm{A}_1\ldots\mathrm{A}_p, i_1\ldots i_r}(z_i), \quad \mathrm{b}_{\mathrm{A}_1\ldots\mathrm{A}_p} = \mathrm{b}_{\max}, \quad r = 0, 1, \ldots, 2\mathrm{b}_{\max}$$

span an invariant subspace of the irreducible representation, and therefore span the representation. QED

The preceding two Propositions combine to give the following.

Theorem III.11 *Suppose that* $\mathbf{T}$ *belongs to an irreducible representation of the Lorentz group. Then, the* q^{th} *order alignment scheme* $\mathcal{A}^q$ *describes the* q^{th}*-order singular points of the top scheme* $\mathcal{A}$*. In other words, every equation of form* (12) *is a linear combination of equations of form* (17)*, and vice versa.*

IV. CLASSIFICATION.

A. Alignment type.

Algebraically special tensors can be characterized in terms of the existence of aligned vectors, with increasing specialization indicated by a higher order of alignment. In a nutshell, one tries to normalize the form of the tensor by choosing $\boldsymbol{\ell}$ and $\boldsymbol{n}$ so as to induce the vanishing of the largest possible number of leading and trailing null-frame scalars. The tensor can then be categorized by the extent to which such a normalization is possible.

Definition IV.1 *Let* $\mathbf{T}$ *be a rank* p *tensor, and let* $\boldsymbol{\ell}$ *be an aligned vector whose order of alignment is as large as possible. We define the* **primary alignment type** *of the tensor to be* $\mathrm{b}_{\max} - \mathrm{b}(\boldsymbol{\ell})$*. If there are no aligned directions, i.e., the alignment equations are over-determined, we will say the alignment type is G,* **general type**.

Supposing that an aligned $\boldsymbol{\ell}$ does exist, we let $\boldsymbol{n}$ be a null-vector of maximal alignment, but subject to the constraint $n^a l_a = 1$.

Definition IV.2 *We define the* **secondary alignment type** *of the tensor to be* $\mathrm{b}_{\max} - \mathrm{b}(\boldsymbol{n})$*, and define the* **alignment type** *of the tensor to be the pair consisting of the primary and the secondary alignment type.*

If $\boldsymbol{\ell}$ is the unique aligned direction, i.e. if no aligned $\boldsymbol{n}$ exists, then we define the alignment type to be the singleton consisting of the primary alignment type. We will also speak of complex and real types, according to whether or not we permit $\boldsymbol{\ell}, \boldsymbol{n}$ to be complex.

By way of example, let us apply the alignment formalism to the description of vectors. Let $\mathbf{v} = v_0\, \boldsymbol{n} + v_1\, \boldsymbol{\ell} + v_i\, \boldsymbol{m}^i$ be a non-zero, real vector. By (4) (11) (12), the zeroth and first order alignment equations are, respectively,

$$\tilde{v}_0 \;=\; v_0 + z^i v_i - \tfrac{1}{2} z^i z_i v_1 = 0, \tag{20}$$
$$\tilde{v}_i \;=\; v_i - z_i v_1 = 0. \tag{21}$$

The covariance of the above equations is easily verified using (16). Also note that, as asserted by Theorem III.11, equations (21) are spanned by the partial derivatives of equation (20).

Rewriting (20) as

$$\delta^{ij}(v_1 z_i - v_i)(v_1 z_j - v_j) = 2 v_0 v_1 + v^i v_i,$$

we see that there are three possibilities, according to the sign of the norm $v^a v_a$. If $\mathbf{v}$ is space-like, i.e., the norm is positive, then $\mathbf{V}(\mathcal{A})$ is a hypersphere. If $\mathbf{v}$ is null, then $\mathbf{V}(\mathcal{A})$ consists of a unique PND of alignment order 1. If $\mathbf{v}$ is time-like, then $\mathbf{V}(\mathcal{A})$ is a complex hypersphere; there are no real aligned directions.

Thus, space-like vectors have alignment type (1,1). This means that every space-like vector can be put into the form $\zeta^i \boldsymbol{m}_i$. Time-like vectors are of real type G, but have complex alignment type (1,1). They can also be normalized to the form $\zeta^i \boldsymbol{m}_i$, but the required null frame and coefficients ζ_i need to be complex. Finally, a null vector $\boldsymbol{k}$, $k^a k_a = 0$ has alignment type (2). This is because a null vector can be normalized to the form $\boldsymbol{\ell}$ by taking a null frame for which $\boldsymbol{\ell} = \boldsymbol{k}$. There is no secondary type, because a null vector can only be aligned with a multiple of itself.

B. Four-dimensional bivectors.

It will be useful, at this point, to consider the covariant classification of bivectors in four-dimensional Minkowski space. The bivector boost weights are shown in (10). Applying (4) (11) we obtain the following form for the alignment equations:

$$\tilde{K}_{02} \;=\; K_{02} - K_{01} z_2 - K_{23} z_3 + \tfrac{1}{2} K_{12}(z_2^2 - z_3^2) + K_{13} z_2 z_3 = 0\,, \tag{22}$$
$$\tilde{K}_{03} \;=\; K_{03} + K_{23} z_2 - K_{01} z_3 - \tfrac{1}{2} K_{13}(z_2^2 - z_3^2) + K_{12} z_2 z_3 = 0\,. \tag{23}$$

According to Theorem III.11, the equations for first order alignment can be obtained by taking partial derivatives of (22)(23):

$$\tilde{K}_{01} \;=\; -\tilde{K}_{02,2} = -\tilde{K}_{03,3} = K_{01} - K_{12} z_2 - K_{13} z_3 = 0\,, \tag{24}$$
$$\tilde{K}_{23} \;=\; -\tilde{K}_{02,3} = \tilde{K}_{03,2} = K_{23} - K_{13} z_2 + K_{12} z_3 = 0\,. \tag{25}$$

Without loss of generality K_{13}, K_{12} are not both zero. By performing a spin and then a boost we can change to a null-frame where $K_{13} = 0, \quad K_{12} = 1$. The top alignment equations (22)(23) may now be written as

$$(z_2 - K_{01})^2 - (z_3 + K_{23})^2 \;=\; (K_{01})^2 - (K_{23})^2 - 2K_{02}\,, \tag{26}$$
$$(z_2 - K_{01})(z_3 + K_{23}) \;=\; -K_{01} K_{23} - K_{03}\,. \tag{27}$$

Generically, these equations have 4 discrete solutions: two real and two complex. These correspond to 2 real and 2 complex principal null directions. Aligning $\boldsymbol{\ell}$ and $\boldsymbol{n}$ gives the canonical form: $K_{ab} = \lambda\, m^2{}_{[a} m^3{}_{b]} + \mu\, \ell_{[a} n_{b]}$. Thus, generically, the alignment type is (1,1). The exceptional case corresponds to

$$(K_{01})^2 - (K_{23})^2 - 2K_{02} = 0, \quad K_{01}K_{23} + K_{03} = 0\,.$$

Here, the complex PNDs merge with the real ones to create a singular, real PND. In this case, the canonical form is $K_{ab} = l_{[a} m^2{}_{b]}$. Such a bivector has alignment type (2).

C. Second-order symmetric tensors.

Let $R_{ab} = R_{ba}$, $R^a{}_a = 0$, be a traceless, second-order symmetric tensor. The maximum boost weight is 2, with R_{00} the scalar of maximum boost weight. Thus, by (4) (11) the unique top alignment equation has the form

$$\tilde{R}_{00} = R_{00} + 2R_{0i}z^i + R_{ij}z^i z^j - 2R_{01}r^2 - 2R_{1j}z^j r^2 + R_{11}r^4 = 0, \tag{28}$$

where we have defined $r^2 = \frac{1}{2} z^i z_i$. The homogeneous form of (28) is

$$R_{ab}\, k^a k^b = 0, \quad k^a k_a = 0. \tag{29}$$

Alignment type, by itself, is insufficient to fully classify symmetric R_{ab}. To obtain a classification, it is necessary to make use of the algebro-geometric properties of the alignment variety defined by equation (28).

This approach is a natural generalization of the Penrose-Rindler method for covariant classification of four-dimensional, symmetric tensors and spinors [6]. In four dimensions, equations (28) (29) describe a certain class of algebraic curves on the 2-sphere. A comprehensive classification of R_{ab} based on the classification of such curves is due to Penrose [12]. See [1, Ch. 5] for other classification approaches and additional references.

For our part, we will describe a generalization of the Penrose classification to arbitrary dimensions. The key technique here is the expression of the alignment equation (29) using the affine form (28). Generically, (28) describes a fourth degree polynomial. However, as soon as one considers singular cases, one can perform a Möbius transformation and reduce (28) to a quadratic polynomial, greatly facilitating the classification. Indeed, the bulk of the classification — the singular cases — is reduced to the problem of classifying quadratic equations in m variables up to Euclidean transformations.

We do this in detail for the 4D case. The higher dimensional classification, while admitting more particular cases, does not materially differ from the four-dimensional case. The canonical forms for the four-dimensional alignment equations are shown in Table II. The multiplicities in the Segre type (third column) are listed in order from lowest to highest eigenvalue. A comma is used to separate an eigenvalue with a time-like eigenvector. The last column shows the alignment type. Where the complex alignment type differs from the real one, the complex type is given first.

The first step in the classification, is to separate the singular and the non-singular cases.

Proposition IV.3 *The alignment variety* (28) *is singular if and only if* $R^a{}_b$ *possesses a multiple eigenvalue.*

	$\tilde{R}_{00}(x,y)$		Segre type	Align. type
1.	$r^4+ax^2+by^2+1$	, $1<a<b$	$111,1$	(1,1)/G
2.	$r^4+ax^2+by^2-1$	, $a\neq b$	$11,Z\overline{Z}$	(1,1)
3.	$r^4+ax^2+by^2+1$	, $a<b<-1$	$11,1,1$	(1,1)
4.	$r^4+2ar^2+1=0$	, $a>1$	$1,1(11)$	(2,2)/G
5.	$r^4+2ar^2+1=0$	, $\lvert a\rvert<1$	$1,(11)1$	(2,2)/G
6.	$r^4+2ar^2+1=0$	, $a<-1$	$(11),1,1$	(2,2)/(1,1)
7.	$r^4+2ar^2-1=0$		$(11),Z\overline{Z}$	(2,2)/(1,1)
8.	$(r^2+1)^2$		$1,(111)$	(2,2)/(2)
9.	$-x^2+ay^2+1$	, $a>0$	121	(2,1)
10.	$-x^2+ay^2+1$	, $a<0$	112	(2,1)
11.	x^2+ay^2+1	, $a>0$	211	(2,1)/(2)
12.	x^2+ay^2	, $a<0$	$1(1,1)1$	(2,2)
13.	x^2+ay^2	, $a>0$	$(1,1)11$	(2,2)
14.	x^2-y		13	(2,1)
15.	$-x^2-y^2+1$		$(11)2$	(2,1)
16.	x^2+y^2+1		$2(11)$	(2,1)/(2)
17.	$-x^2+1$		$1(12)$	(2,1)
18.	x^2+1		$(12)1$	(2,1)/(2)
19.	x^2+y^2		$(11)(11)$	(2,2)
20.	x^2		$(11,1)1$	(2,2)
21.	x		(13)	(3,1)
22.	1		(112)	(4)
23.	0			

TABLE II: Canonical alignment equations for R_{ab} in 4D

Lemma IV.4 *The linear transformation $R^a{}_b$ has a multiple eigenvalue if and only if it possesses a real or a complex null eigenvector.*

Proof. Suppose that $\boldsymbol{n}$ is a null eigenvector with eigenvalue λ. We complete to a null frame $\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_i$, and note that $R_{11} = R_{1i} = 0$, and that $R_{01} = \lambda$. Let us consider two cases, according to whether λ is an eigenvalue of the matrix $R^i{}_j$. If yes, let $\mathbf{v} = v^i \boldsymbol{m}_i$ be an eigenvector. Hence,

$$R^a{}_b\, v^b = \lambda v^a + R_{0i} v^i n^a,$$

which proves that λ is a multiple eigenvalue. In the opposite case, the matrix $R^i{}_j - R_{01}\delta^i{}_j$ is non-singular, and hence by performing a null-rotation about $\boldsymbol{n}$ (equivalently, a translation in the z_i) we can switch to a null frame where $R_{0i} = 0$. Now, we have

$$R^a{}_b\, \ell^b = \lambda \ell^a + R_{00}\, n^a,$$

which again shows that λ is a multiple eigenvalue.

Conversely, suppose that $R^a{}_b$ has a multiple eigenvalue λ. Hence there exist vectors $\mathbf{u}, \mathbf{v}$

such that

$$\begin{aligned} R^a{}_b\, u^b &= \lambda u^a, \\ R^a{}_b\, v^b &= \lambda v^a + \kappa u^a. \end{aligned}$$

Consider the case $\kappa \neq 0$. We have

$$R_{ab} u^a v^b = \lambda\, u^a v_a = \lambda\, v^a u_a + \kappa\, u^a u_a,$$

and hence $u^a u_a = 0$, as desired. Finally, suppose that $\kappa = 0$. If either $\mathbf{u}$ or $\mathbf{v}$ are null eigenvectors, we are done. If neither is a null-vector, we construct a null eigenvector (possibly complex) of the form $\mathbf{u} + \alpha\mathbf{v}$, where α is a solution of the following quadratic equation:

$$u^a u_a + 2u^a v_a\, \alpha + v^a v_a\, \alpha^2 = 0.$$

QED

Proof of proposition IV.3. Suppose that (28) is singular. Choose an $\boldsymbol{\ell}$ that spans a singular direction, and complete to a null frame $\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_i$. Note that if the singular direction is complex, then the null frame will have to be complex also. By Theorem III.11 we have $R_{00} = R_{0i} = 0$, and hence $R^a{}_b\, \ell^b = R_{01} \ell^a$. Hence, $\boldsymbol{\ell}$ is a null eigenvector and, by the Lemma, $R^a{}_b$ has multiple eigenvalues. Conversely, suppose that $R^a{}_b$ has multiple eigenvalues. By the lemma, we can choose a null eigenvector $\boldsymbol{\ell}$. Completing, to a null frame, we obtain $R_{00} = R_{0i} = 0$, which proves that $\boldsymbol{\ell}$ spans a singular direction. QED

Turning first to the non-singular case, by the above Proposition all eigenvalues are distinct, and hence we can choose an orthonormal set of $m = n - 2$ space-like eigenvectors $\boldsymbol{m}_i$ with eigenvalues λ_i. We complete these to a null-frame, and perform a boost so that $R_{00} = \pm R_{11}$. In this way the alignment equation simplifies to the following canonical form

$$r^4 + \sum_i a_i {z_i}^2 \pm 1 = 0, \qquad a_i = \frac{\lambda_i - R_{01}}{R_{11}}. \tag{30}$$

We now distinguish two sub-cases, depending on the sign in the above equation. If the sign is negative, the alignment variety is homeomorphic to the sphere $\mathbb{S}^{m-1}$. One can show that this case is distinguished by the presence of a pair of conjugate complex eigenvalues of $R^a{}_b$.

If the sign in (30) is positive, the topology of the real alignment variety is determined by the position of the eigenvalue corresponding to the time-like eigenvector. Let p be the number of times when $a_i < -1$. If $p > 0$, then the real part of the alignment variety is homeomorphic to a product of spheres, $\mathbb{S}^{p-1} \times \mathbb{S}^{m-p}$, where we define the zero-dimensional sphere $\mathbb{S}^0$ to be the set consisting of two distinct points. If $p = 0$, the alignment variety has no real points. In all these cases, the eigenvalues of $R^a{}_b$ are real, with exactly one eigenvalue corresponding to a time-like eigenvector.

In 4D, the non-singular cases are described by lines 1, 2, 3 of Table II. In case 1, the alignment variety has no real points. For cases 2 and 3, the alignment variety is homeomorphic to $\mathbb{S}^1$ and $\mathbb{S}^1 \times \mathbb{S}^1$, respectively.

In the degenerate cases, Lemma IV.4 shows that there exists at least one (possibly complex) null eigenvector. The degenerate cases are, therefore, divided according to whether the null eigenvector of $R^a{}_b$ is real or complex. If the null eigenvector is complex, then the canonical form of the alignment equation is (30), with $a_i = a_j$ for some $i \neq j$. In 4D, the singular cases with a complex eigenvector correspond to lines 4-8 of Table II.

If there exists a real null eigenvector, we use a null frame $\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_i$ with $\boldsymbol{n}$ spanning the singular direction. In this way $R_{11} = R_{1i} = 0$, and (28) reduces to a quadratic equation. The subgroup of Möbius transformations preserving $[\boldsymbol{n}]$ is precisely the group of Euclidean similarity transformations: rotations, translations, and scalings. Thus, in the real, singular case the classification reduces to the familiar problem of classifying conics up to Euclidean transformations. In 4D, these cases correspond to lines 9-23 of Table II.

D. Weyl-like tensors.

We define a *Weyl-like tensor* C_{abcd} to be a traceless, valence 4 tensor with the well-known index symmetries of the Riemann curvature tensor, i.e.,

$$C_{abcd} = -C_{bacd} = C_{cdab}, \quad C_{abcd} + C_{acdb} + C_{adbc} = 0, \quad C_{abc}{}^{b} = 0.$$

We let $\mathcal{W}_n$ denote the vector space of n-dimensional Weyl-like tensors. It isn't hard to show that $\mathcal{W}_n$ has dimension $\frac{1}{12}(n+2)(n+1)n(n-3)$.

TABLE III: Boost weight of the Weyl scalars.

2	1	0	-1	-2
C_{0i0j}	C_{010i}, C_{0ijk}	$C_{0101}, C_{01ij}, C_{0i1j}, C_{ijkl}$	C_{011i}, C_{1ijk}	C_{1i1j}

The maximal boost weight for a Weyl tensor is given by $\mathrm{b}_{\max} = 2$. There are $\frac{1}{2}n(n-3)$ independent scalars of maximal boost weight. We define the Weyl alignment equations to be the top alignment equations

$$\tilde{C}_{0i0j} = 0, \tag{31}$$

where the left-hand side is defined in (11). This is a system of $\frac{1}{2}n(n-3)$, fourth order equations in $n-2$ variables.

In 4D, the principal null directions of the Weyl-like tensor are defined in terms of the so-called PND equation[1, 7]:

$$k^b k_{[e} C_{a]bc[d} k_{f]} k^c = 0, \quad k^a k_a = 0. \tag{32}$$

Let us show that, in all dimensions, the above system of equations is equivalent to the Weyl alignment equations (31)

Proposition IV.5 *For every dimension n, a null vector $\boldsymbol{k}$ satisfies* (32) *if and only if $\boldsymbol{k}$ is aligned with C_{abcd}.*

Proof. Let a null vector $\boldsymbol{k}$ be given. Completing to a null-frame $\boldsymbol{\ell} = \boldsymbol{k}, \boldsymbol{n}, \boldsymbol{m}_i$, we have

$$k^b C_{abcd} k^c = -C_{0i0j} m^i{}_a m^j{}_d - 2C_{010i} \ell_{(a} m^i{}_{b)} - C_{0101} \ell_a \ell_b,$$
$$k^b k_{[e} C_{a]bc[d} k_{f]} k^c = -C_{0i0j}\, \ell_{[e} m^i{}_{a]} \ell_{[f} m^j{}_{d]}$$

Hence, (32) holds if and only if $C_{0i0j} = 0$, i.e., all scalars of boost weight 2 vanish. This is precisely the definition of $\boldsymbol{k}$ being aligned with C_{abcd}. QED

Henceforth, let us set $m = n-2$. For $n \geq 4$ we have

$$\tfrac{1}{2}(m+2)(m-1) \geq m,$$

with equality if and only if $n = 4$. Thus, a four-dimensional Weyl-like tensor always possesses at least one aligned direction (see below). The situation in higher dimensions is described by the following:

Theorem IV.6 *If $n \geq 5$, then the subset of $\mathcal{W}_n$ with complex alignment type G is a dense, open subset of $\mathcal{W}_n$. In other words, the generic Weyl-like tensor in higher dimensions does not possess any aligned null directions (not even complex ones).*

Proof. Let C_{abcd} be a Weyl-like tensor with scalars C_{ABCD} relative to some fixed null frame. Let C'_{abcd} be a Weyl-like tensor of boost order 1 defined by

$$C'_{\mathrm{ABCD}} = \begin{cases} C_{\mathrm{ABCD}} & \text{for } \mathrm{b}_{\mathrm{ABCD}} \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$\tilde{C}_{0i0j}(z_k) = C_{0i0j} + \tilde{C}'_{0i0j}(z_k).$$

Next, let us define the mapping $\gamma : \mathbb{C}^m \to \mathbb{C}^{\frac{1}{2}m(m+1)}$ by

$$\gamma_{ij}(z_k) = -\tilde{C}'_{0i0j}(z_k), \quad i \leq j.$$

Hence, C_{abcd} possesses a non-infinite aligned direction if and only if C_{0i0j} is in the image of γ. However, generically, the image of the mapping γ is an affine subvariety of dimension m, whereas the set of all C_{0i0j} satisfying $C_{0i0}{}^{i} = 0$ is a subspace of dimension $\frac{1}{2}(m+2)(m-1)$. Therefore, the set of all C_{abcd} with an aligned null direction is an m-dimensional subvariety of the affine space of all Weyl-like tensors. The desired conclusion follows immediately. QED

Let us now re-derive the well-known Petrov classification of 4-dimensional Weyl-like tensors in terms of alignment type. The calculations are facilitated by the use of the NP tetrad[6], and so we introduce the following notation:

$$\begin{aligned} z_\pm &= \frac{1}{\sqrt{2}}(z_2 \pm i z_3), \qquad \partial_\pm = \frac{1}{\sqrt{2}}\left(\partial_2 \mp i\partial_3\right), \qquad \boldsymbol{m}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{m}_2 \mp \boldsymbol{m}_3) \\ \Psi &= \tilde{C}_{0202} + i\tilde{C}_{0203}, \qquad \Lambda^{\mathrm{A}}{}_\pm = \frac{1}{\sqrt{2}}\left(\Lambda^{\mathrm{A}}{}_2 \pm i\Lambda^{\mathrm{A}}{}_3\right), \end{aligned}$$

where, $\tilde{C}_{\mathrm{ABCD}}$ is given by (11), and where, as per (4), we have

$$\Lambda^{\mathrm{A}}{}_{\mathrm{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -z_+z_- & 1 & z_2 & z_3 \\ -z_2 & 0 & 1 & 0 \\ -z_3 & 0 & 0 & 1 \end{pmatrix}$$

Rewriting (18) we have that

$$\Lambda^{\mathrm{A}}{}_{0,\pm} = -\Lambda^{\mathrm{A}}{}_\pm, \quad \Lambda^{\mathrm{A}}{}_{\pm,\pm} = \Lambda^{\mathrm{A}}{}_1, \quad \Lambda^{\mathrm{A}}{}_{\pm,\mp} = \Lambda^{\mathrm{A}}{}_{1,\pm} = 0.$$

It follows immediately that $\Psi_{,-} = 0$. Hence,

$$\Psi(z_+, z_-) = \Psi(z_+) = (C_{1212} + iC_{1213})(z_+ - \zeta_1)(z_+ - \zeta_2)(z_+ - \zeta_3)(z_+ - \zeta_4)$$

is a fourth degree polynomial of one complex variable. Rewriting the top alignment equations as $\Psi(z_+) = \overline{\Psi}(z_-) = 0$, we deduce that, generically, there are 16 solutions:

$$z_2 = \frac{1}{\sqrt{2}}(\zeta_M + \bar{\zeta}_N), \quad z_3 = \frac{-i}{\sqrt{2}}(\zeta_M - \bar{\zeta}_N), \quad M, N = 1, 2, 3, 4.$$

The 4 real solutions correspond to the case $M = N$.

By Theorem III.11 (this can also be verified directly) the equations for higher order alignment are given by the derivatives

$$\Psi^{(r)}(z_+) = 0, \quad \overline{\Psi^{(r)}}(z_-) = 0, \quad r = 0, \ldots, q, \quad q = 1, 2, 3.$$

It follows that the equations for alignment order q have a solution if and only if $\Psi(z_+)$ possesses a root of multiplicity $q + 1$ or more. In this way, we recover the usual Petrov classification, which counts the root multiplicities of the polynomial $\Psi(z_+)$ — see Table IV.

TABLE IV: The 4D Petrov classification in terms of alignment type.

Weyl type	I	II	D	III	N	O
Alignment type	(1,1)	(2,1)	(2,2)	(3,1)	(4)	(5)
root multiplicities of Ψ	1,1,1,1	2,1,1,1	2,2	3,1	4	

TABLE V: Higher dimensional Weyl types.

Weyl type	G	I	I_i	II	II_i	D	III	III_i	N	O
Alignment type	G	(1)	(1,1)	(2)	(2,1)	(2,2)	(3)	(3,1)	(4)	(5)

Table V lists the possible alignment types of a higher dimensional Weyl-like tensor, grouped into categories which are compatible with the usual Petrov classification. By Theorem IV.6, in 5 dimensions and higher, the generic Weyl-like tensor does not have any aligned directions. Thus, unlike the case in 4D, type I tensors are an algebraically special category.

Another key difference in the higher-dimensional classification is the significance of the secondary alignment type. The proof of Theorem IV.6 is easily adapted to show that, for types I, II, and III, even though an aligned $\boldsymbol{\ell}$ exists, there does not, generically, exist an aligned $\boldsymbol{n}$. This is in contrast to 4 dimensions, where an aligned $\boldsymbol{n}$ can always be chosen. Hence, in higher dimensions we must distinguish types $\mathrm{I}_i, \mathrm{II}_i, \mathrm{III}_i$ as the algebraically special subclasses of types I, II, and III that possess an aligned $\boldsymbol{n}$. Type II_{ii} is a further specialization of type II_i, one where $\boldsymbol{n}$ has alignment order 1. It is denoted by type D in analogy with the 4-dimensional classification. As expected, type O is the maximally degenerate type corresponding to the vanishing of the Weyl-like tensor.

Also, unlike the 4-dimensional classification, the above categories are coarse in the sense that they collect together a number of inequivalent tensor types. Thus, in the 4D classification each alignment type admits a canonical form, but this is not always true for $n > 4$. Section V contains some additional remarks on the classification of curvature tensors in higher-dimensional Lorentz spaces.

V. DISCUSSION

The present paper develops the theory of alignment in Lorentzian geometry. We have defined and studied the general notions of *aligned null direction*, *order of alignment*, and *alignment type*, and applied these ideas to the problem of tensor classification. In particular, we have argued that it is possible to categorize algebraically special tensors in terms of their alignment type, with increasing specialization indicated by a higher order of alignment.

Alignment type suffices for the classification of 4-dimensional bivectors and Weyl-like tensors, but is not sufficiently refined for the classification of second-order symmetric tensors and the classification of tensors in higher dimensions. Thus, the present paper should be considered as a necessary first step in the investigation of covariant tensor properties based on the notion of alignment.

Of particular interest are classes of tensors corresponding to irreducible representations of the Lorentz group, because by Theorem III.11 the higher alignment equations describe the singularities of the top alignment variety. Thus, for irreducible representations the classification problem is reduced to the study of the corresponding moduli space — the variety of all top alignment schemes quotiented by the group of Möbius transformations. The algebraically special classes correspond to the various singular strata of this space.

The representation of Weyl-like tensors is of particular interest, both from the theoretical standpoint and because of physical applications. In higher dimensions, classification needs to go beyond alignment type, and to consider other covariant properties of the Weyl aligned null directions (WANDs). The key concepts and theoretical issues underlying such a classification are sketched out in [13], and are also summarized below.

1. Orthogonal reducibility. Let $\boldsymbol{m}_{\mathrm{A}}$ be a null-frame. Let us fix a $k = 2, \ldots, n-1$, and partition the null frame into a null subframe $\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_2, \ldots, \boldsymbol{m}_{k-1}$ and a positive-definite subframe $\boldsymbol{m}_k, \ldots, \boldsymbol{m}_{n-1}$. Having done this, it will be convenient to call indices $\mathrm{A} = 0, 1, \ldots, k-1$ Lorentzian, and call indices $\mathrm{A} = k, \ldots, n-1$ Riemannian. Let C_{abcd} be a Weyl-like tensor. Scalars C_{ABCD} can now be categorized as being Lorentzian, Riemannian, or mixed, according to whether the indices are all of one type or the other, or whether there are some indices of both types. We will say that the frame $\boldsymbol{m}_{\mathrm{A}}$ defines an *orthogonal decomposition* and that C_{abcd} is *orthogonally reducible* if for some $k = 2, \ldots, n-1$ all mixed scalars vanish.

Orthogonal reducibility is a covariant criterion, and refines the alignment type categories. Indeed, suppose that C_{abcd} is reducible, and let $\breve{C}_{\mathrm{ABCD}}$ denote the k-dimensional reduction obtained by restricting to Lorentz indices. We can then define the *reduced alignment type* of C_{abcd} to be the alignment type of the reduced tensor. Uniqueness of the orthogonal decomposition now becomes a key theoretical question. A closely related question is the extent to which the alignment equations of a reducible tensor decouple into separate equations for the Lorentzian and the Riemannian variables.

Let us also note that non-trivial Weyl-like tensors do not exist if $n < 4$. Thus, reducibility becomes an issue only in dimensions $n \geq 5$, and even then, one of the complementary summands must necessarily vanish for $n \leq 7$. True decomposability can only occur for $n \geq 8$.

2. Cardinality of aligned null directions. For dimension $n = 4$ there are exactly four WANDs, counting multiplicities. For $n \geq 5$, the situation is more complicated. If all WANDs are principal, what is an upper bound for each dimension n? It is also possible to have an infinity of WANDs, as in the case of an orthogonally reducible tensor with vanishing

Riemannian components. When does this type of degeneracy occur, or conversely, what is required for WANDs to be principal?

3. Canonical forms. By definition, classification according to alignment type corresponds to a normal form where the components of leading and trailing boost weight vanish. It would be desirable to refine the classification to the point of a true canonical form. This is possible in the case of type N Weyl-like tensors, which can always be put into the form $\sum_i \lambda_i \ell_{\{a} m^i{}_b \, \ell_c m^i{}_{d\}}$. For type III and III_i tensors, the generic situation is $C_{011i} \neq 0$, and thus the tensor defines a preferred null plane spanned by $C_{011i}\boldsymbol{m}^i$ and $\boldsymbol{\ell}$. One can therefore put the tensor into canonical form by performing a spin and a boost so that $C_{0112} = 1$ and $C_{011i} = 0$ for all $i = 3, 4, \ldots$.

4. Riemann-like tensors. Much of the analysis for higher dimensional tensors can be applied directly to the classification of higher dimensional Riemann curvature tensors. In particular the higher-dimensional alignment types shown in Table V give well defined categories for the Riemann tensor. However, in the case of the Riemann tensor there are additional constraints coming from the extra non-vanishing components. For example, a type I Riemann curvature tensor must satisfy the $\frac{1}{2}m(m+1)$ constraints $R_{0i0j} = 0$, whereas a type I Weyl-like tensor has one less constraint owing to the fact that $C_{0i0}{}^i = 0$.

5. Differential-geometric considerations. The focus of the present paper has been purely algebraic. We have been studying tensors rather than tensor fields, or to put it another way we have been considering tensors at single point of a Lorentzian manifold. There is however, a rich interplay between algebraic type and the differential Bianchi identities.

In 4 dimensions, there are a number of consequences, such as the Goldberg-Sachs theorem and its non-vacuum generalizations [1, 6, 14]. In higher dimensions, differential consequences of the Bianchi identities in type III and N spacetimes have been considered in [15].

In 4D it is possible to use the Bianchi and Ricci equations to construct many algebraically special solutions of Einstein's field equations. The hope is that it is possible to do a similar thing in higher dimensions, at least for the simplest algebraically special spacetimes. The vast majority of today's known higher-dimensional exact solutions are simple generalizations of 4D solutions. This approach may lead to new, genuinely higher dimensional exact solutions. Type N and D solutions may be of particular physical interest. The Goldberg-Sachs type theorems are very useful for constructing algebraically special exact solutions. At present it is unclear to what extent such theorems may be generalized for higher dimensions.

One further application is the classification of VSI manifolds in higher dimensions. The primary alignment type of such manifolds would have to be III, N, or O, while the local coordinate expression for the metric would be a higher-dimensional generalization of the Kundt form[1, 3, 16].

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